A GRAPH-THEORETIC PROOF OF THE Δ-SYSTEM LEMMA

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A graph is said to be *locally countable* if every vertex has at most countably-many neighbors.

Observation 1. *A graph is locally countable if and only if each of its connected components is countable.*

For a graph G on a set V, a subset $I \subseteq V$ is called *independent* if no two vertices in it are adjacent.

Lemma 2 (Uses AC)**.** *A locally countable graph G on an uncountable set V admits an independent set $I \subseteq V$ with $|I| = |V|$.*

Proof. Each connected component is countable, so their collection has cardinality $|V|$. Choosing a point from each connected component gives a desired independent set. □

The *intersection graph* on any family V of sets is defined by putting an edge between distinct $u, v \in V$ exactly when $u \cap v \neq \emptyset$. Call a family V of sets *point-locally countable* if for each $a \in \cup V$, the set $V_a := \{v \in V : a \in v\}$ is countable.

Lemma 3 (Uses Countable-AC)**.** *The intersection graph on any point-locally countable family V of countable sets is locally countable.*

Proof. For each $v \in V$, each $a \in v$ is contained in at most countably-many other vertices in V. Since v is countable and countable union of countable sets is countable, it intersects at most countably-many other sets in V. □

Lemmas 2 and 3 give the following tongue twister:

Proposition 4 (Uses AC)**.** *The intersection graph on a point-locally countable uncountable family V of countable sets admits a pairwise disjoint collection $I \subseteq V$ with $|I| = |V|$.*

Corollary 5 (The Δ-system Lemma, Uses AC)**.** *Every uncountable family V of finite sets contains an uncountable Δ-system.*

Proof. By shrinking V using uncountable-to-countable Pigeonhole Principle (involves AC), we may assume that for some $n \in \mathbb{N}$, all sets in V have the same size n and we prove the statement by induction on n. The base case $n = 1$ is trivial, so suppose the statement is true for $n - 1 \geqslant 1$ and let V be an uncountable family of sets of size n.

If V is point-locally countable, Proposition 4 finishes the proof, so suppose that V is not point-locally countable and let $a \in \cup V$ be such that the set V_a is uncountable. Removing a from the sets in V_a, applying induction, and putting a back yields an uncountable Δ-system $V'_a \subseteq V_a$. □